# Scaling Exponents of Strong Turbulence in the Eddy Viscosity Approximation

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February 5, 2008

**Abstract**

A dynamic equation for velocity structure functions $S_n(r) =< (u(x) - u(x'))^n >$ in strong turbulence is derived in the one-loop (eddy viscosity) approximation. This homogemeous differential equation yields scaling exponents $\xi_n$ in the relations $S_n(r) \propto r^{\xi_n}$ which are in a very good agreement with experimental data.

Kolmogorov's (1941) relation for the third-order structure function which is in the limit of zero viscosity $\nu_0 \rightarrow 0$,

$$S_3 =< (u(2) - u(1))^3 >= -\frac{4}{5}\mathcal{E}r \tag{1}$$

where $\mathcal{E} = \nu\overline{(\partial_i u_j)^2}$ is the mean dissipation rate of turbulent kinetic energy, $u(i) \equiv u(\mathbf{x_i})$ is the value of the $x$-component of the velocity field at the point $\mathbf{x}$, and $r = \mathbf{n} \cdot (\mathbf{x_2} - \mathbf{x_1})$ is the value of the displacement along the $x$-axis ($\mathbf{n}$ is the unit vector in the $x$-direction) [1]. Applying dimensional considerations to this dynamic relation, Kolmogorov also made the prediction:

$$S_2 =< (u(2) - u(1))^2 >\propto \mathcal{E}^{\frac{2}{3}} r^{\frac{2}{3}} \tag{2}$$

leading to the celebrated Kolmogorov energy spectrum:

$$E(k) = C_K \mathcal{E}^{\frac{2}{3}} k^{-\xi} \tag{3}$$

with $\xi = 5/3$. Kolmogorov went even further by generalizing (2) to the structure function $S_n(r)$ of arbitrary order $n$:

$$S_n(r) =< (u(2) - u(1))^n >= A_n(\mathcal{E} r)^{\frac{n}{3}} \tag{4}$$

Experimental investigations have supported the relation (2) with good accuracy, giving for the exponent of the second-order structure functions $\xi_2 \approx 1.66 - 1.75$. At the same time substantial deviations of $\xi_n$ from the K41 values $\xi_n = n/3$ from $n > 3$ have also been observed. It is interesting that no data on the scaling exponents of the structure functions $S_n$ with $n < 1$ have yet been reported.

Ever since Kolmogorov's work, dimensional considerations were the most dominant way of evaluating the scaling exponents. This is why until recently no progress in understanding the intermittency of strong turbulence was achieved. In 1994 R. H. Kraichnan, considering the problem of a passive scalar advection in a rapidly- changing- in- time random velocity field, realized that anomalies, coming from the dissipation terms in the equation of motion, result in a homogeneous differential equation for $S_n(r)$, leading to an algebraic form of the scalar structure functions $S_n(r) \propto r^{\xi_n}$ with scaling exponents $\xi_n$ determined by the numerical values of the coefficients in the equation [2]. This is why the scaling exponents $\xi_n$ cannot be obtained on the basis of dimensional considerations. Later, groundbreaking works by Gawedzkii and Kupiainen [3 ], Chertkov et. al. [4 ] proved that in the vicinity of the gaussian limits the scaling of scalar structure functions is determined by the zero modes of the homogeneous differential equations for $S_n(r)$, which for this problem can be written explicitly [5 ], [ 6 ]. The results of Refs. [ 3]- [4 ] have been confirmed in ref. [7 ] using a different approach. At about the same time Shraiman and Siggia [8 ], considering a model of a passive scalar based on the concept of eddy diffusivity, showed that zero modes play the most important part in determination of the scaling exponents of structure functions.

Since Richardson's (1926) work, effective transport coefficients (eddy viscosity and eddy diffusivity), though not rigorously justified, have been used for the quantitative description of transport phenomena in strongly turbulent engineering flows. It is safe to state that by now these concepts have evolved into an engineering tool widely used for design purposes throughout mechanical engineering. The idea behind the method is simple [9], [10]. Consider the Navier-Stokes equations for an incompressible fluid:

$$u_{it} + \mathbf{u} \cdot \nabla u_i = S_i - \nabla_i p + \nu_0 \nabla^2 u_i \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0$$

where $\mathbf{S}(\mathbf{x}, \mathbf{t})$ $(\nabla \cdot \mathbf{S} = 0)$ is a large-scale source function. Using the incompressibility condition the pressure $p$ can be expressed in terms of the velocity field. It is useful to write the equation for the Fourier-transform $\mathbf{u}(\mathbf{k}, \omega)$:

$$-i\omega u_l(\mathbf{k}, \omega) + \frac{i}{2} P_{lmn}(\mathbf{k}) \int d\mathbf{q} d\Omega u_m(\mathbf{q}, \omega) u_n(\mathbf{k} - \mathbf{q}, \omega - \Omega) = S_l(\mathbf{k}, \omega) - \nu_0 k^2 u_l(\mathbf{k}, \omega)$$

where the projection operator $P_{lmn}(\mathbf{k})$ is:

$$P_{lmn}(\mathbf{k}) = k_m P_{ln}(\mathbf{k}) + k_n P_{lm}(\mathbf{k})$$

and $P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$. In the eddy viscosity approximation the effects of the small scales $(q >> k)$ on the modes $\mathbf{u}(\mathbf{k})$ is represented in terms of the eddy viscosity [10 ] (we neglect the small eddy noise contribution):

$$-i\omega u_l(\mathbf{k}, \omega) + \frac{i}{2} P_{lmn}(\mathbf{k}) \int d\mathbf{q} d\Omega u_m(\mathbf{q}, \omega) u_n(\mathbf{k} - \mathbf{q}, \omega - \Omega) \approx S_l - \Gamma k^{2+a} u_l(\mathbf{k}, \omega) - \nu_0 k^2 u_l(\mathbf{k}, \omega) \tag{6}$$

where

$$a = -\frac{2 + \xi_2}{2} \tag{7}$$

The integration is carried out over the interval: $0 < q < k$; $-\infty < \omega < \infty$. This means that this equation describes the velocity field averaged over small-scale fluctuations with $q > k$. This result must be considered as a general model for the velocity field $u_l(\mathbf{k},\omega)$ where the effects of the small- scale fluctuations are accounted for by the eddy viscosity with $\nu(k) \propto \Gamma k^a$. In principle, this equation must be solved subject to initial and boundary conditions for each mode $u(\mathbf{k},t)$. A similar model has recently been used for evaluation of the anomalous scaling in the problem of a passive scalar advected by a random velocity field by Shraiman and Siggia [8 ]. The role of the large scale motions in the dynamics of the small-scale velocity fluctuations, described by the non-linear contribution to (6), is two-fold. First, the large-scale structures kinematically transfer ( sweep ) the small-scale fluctuations. This process does not lead to the energy redistribution. Secondly, the large- scale motions serve as an energy source for the small-scale dynamics which can be accounted for by the random forcing function $\mathbf{f}$ which is a complex functional of the velocity field. Eventually, we will be interested in the equation of motion for the structure functions of velocity differences $\Delta u = u(2) - u(1)$, for which sweeping is not too important. This allows us to write (6) as a simple Langevin-like equation for $\mathbf{u}(\mathbf{k})$:

$$-i\omega u_l(\mathbf{k},\omega) \approx S_l + f_l - \Gamma k^{2+a} u_l(\mathbf{k},\omega) - \nu_0 k^2 u_l(\mathbf{k},\omega) \tag{8}$$

In this approximation the equation for the structure function $S_n(r)$ can be written readily in the three-dimensional case:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^{2-a} \frac{\partial S_n}{\partial r} = nD \tag{9}$$

where

$$D =< (\Delta S + \Delta f + \nu_0(\nabla_2^2 u(2) - \nabla_1^2 u(1)))(\Delta u)^{n-1} >$$

We are interested in the behaviour of the structure functions for small values $r/L << 1$. Since the source $S$ is assumed to act at large scales only we neglect it in what follows. As in the problem of a passive scalar, the derivation of the $D$-term is a difficult task. Here

the problem is even harder since we do not know much about the effective energy source $\mathbf{f}$. It is clear, however, that the eddy viscosity approximation can be accurate only for $S_n(r)$ with the relatively small $n > 0$. Thus, the values of $S_n$ are dominated by the part of the probability density $P(\Delta u, r)$ where $\Delta u \approx (\Delta u)_{rms}$. We assume:

$$D = f(r) S_n(r) \tag{10}$$

with $f(r)$ independent on $n$. A more detailed argument leading to (10) will be presented below where it will be shown that (10) is consistent with the eddy viscosity approximation. The expression (10) is not dissimilar to the anzatz introduced by Kraichnan in his theory of a passive scalar [2]. The function $f(r)$ is fixed by the relation (1). Introducing the new variable

$$r = \frac{4}{5}\mathcal{E}|x_1 - x_2| \tag{11}$$

so that

$$S_3(r) = r \tag{12}$$

we have

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^{2-a}\frac{\partial S_n}{\partial r} = \frac{n}{3}(2-a) r^{-2-a} S_n \tag{13}$$

One can see that the relation (12) is satisfied by this equation. The corresponding equation for the probability density $P(\Delta u, r) \equiv P(U, r)$ is:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^{2-a}\frac{\partial P(U,r)}{\partial r} = -\frac{(2-a)}{3} r^{-2-a}\frac{\partial}{\partial U} U \; P(U,r)$$

The equation for the exponents $\xi_n$ following from (13) is:

$$\xi_n\left(1 + \frac{2+\xi_2}{2} + \xi_n\right) - \frac{n}{3}\left(2 + \frac{2+\xi_2}{2}\right) = 0 \tag{14}$$

.

Thus, we see that all exponents $\xi_n$ are determined in terms of $\xi_2$. Substituting $n = 2$ into (14) we derive $\xi_2 = 0.7257$. Using this value we have:

$$\xi_n \approx -1.1815 \ + \ (1.3958 + 1.1210\ n)^{\frac{1}{2}} \tag{15}$$

This formula gives the following values for the exponents: $\xi_{1/4} \approx 0.1131$; $\xi_{1/2} \approx 0.2171$; $\xi_{3/4} \approx 0.3140$; $\xi_1 \approx 0.4046$; $\xi_2 = 0.7257$; $\xi_4 = 1.2433$; $\xi_5 \approx 1.4644$; $\xi_6 \approx 1.6684$. It is interesting that, strictly speaking, the scalings of all moments, including those with $n < 1$, are anomalous and cannot be obtained from dimensional considerations. It is possible to formally continue the expression (15) into the interval $n < 0$. We can see that the moments $S_n$ with $n < n_c = -\frac{1.3958}{1.1210} \approx -1.2451$ do not exist. Since the PDF $P(\Delta u, r) \neq 0$ at the origin $\Delta u = 0$, it is expected that the moments $S_n(r)$ with $n < -1$ diverge. Thus, the fact that the relation (15) gives the critical moment-order $n_c \approx -1.24$, close to $n_c = -1$, indicate that formula (15) might give close-to- correct values of the scaling exponents $\xi_n$ for $n > n_c$, which are not too far from critical $n_c = -1$. We have: $\xi_{-1/4} \approx -0.126$; $\xi_{-1/2} \approx -0.269$; $\xi_{-3/4} \approx -0.438$; $\xi_{-1} \approx -0.660$ .

Let us now consider the expression for the dissipation term $D$ which is proportional to $\nu_0 \rightarrow 0$. The eddy viscosity approximation limits the renormalized perturbation expansion in powers of the non-linearity by one-loop contributions only. The technical details of the procedure vere described in Refs. [10], [11], [12]. We are interested in the correlation function, similar to $D$:

$$D_1 = \nu_0 \ < u(q_1)....q_i^2 u_(q_i)...u(k - q_1.... - q_{n-1}) > \tag{16}$$

where $i = 1;\ 2; .....n - 1$; there is one contribution involving $q_n^2 = (k - q_1.... - q_{n-1})^2$. The expansion is generated by the iteration procedure involving the Navier-Stokes equations (5) or (6) (see Refs. [10], [11]). In the zero -loop approximation we have

$$D_1^0 \approx n \mathcal{E} S_{n-2}(r)$$

where the $O(1)$ dissipation rate $\mathcal{E} = \nu_0 \int k^2 E(k) dk$. This term is obtained from (16) in the limit of the "free momentum" $k \rightarrow 0$. Substituting this expression into the equation of motion

(9) we find that, for the solution found above, balance is impossible since $-1+\frac{\xi_2}{2}+\xi_n \neq \xi_{n-2}$. Moreover, it is easy to see that the zero-loop contributions to the expression for $D$, we are interested in, cancel in the limit $k \to 0$. In the one-loop approximation we can have $n$ u.v.-divergent contributions of the order $\nu_0 S_n k_d^2$ and only one term coming from the $k$-dependent $q_n$ in relation (16) (here $k$ is a free "momentum"):

$$D_1^2 \propto \nu(r)\frac{dS_n}{dr^2} \propto \frac{\nu(r)S_n}{r^2} \tag{17}$$

where the effective (eddy) viscosity $\nu(r) = O(r^{1+\frac{\xi_2}{2}})$. Substituting (17) into (9) and fixing the proportionality coefficient to satisfy Kolmogorov's 4/5 law (1) we derive the equation (13) used for evaluation of the exponents $\xi_n$. The u.v.-divergent contributions are assumed to cancel for the eddy viscosity approximation to work. This fact was proved to be correct for the model case of a simple effective forcing function in Ref. [12]. It was shown that this cancellation is the result of overall energy balance that requiers $< \mathbf{S} \cdot \mathbf{u} >= \mathcal{E}$. In general, the rigorous demonstration that it is so is a difficult task. All we can say now is that the right side of equation (13) is consistent with the eddy viscosity approximation.

The fact that eddy viscosity works so well for the description of quite complex engineering flows was known for many years in the engineering community. For these many years, the eddy viscosity concept was treated with suspicion or even contempt by many physisists due to the long- standing belief that, being a result of one- loop closure, it cannot be useful for the description of the "non-perturbative" intermittency of turbulence. Recently, it has been shown in direct numerical experiments on the two-time, two-point correlation function in one-dimensional Kolmogorov turbulence for a forced Burgers equation that eddy viscosity works well to descibe the small scale dynamics in this complex and strongly intermittent system [13]. The work of Shraiman and Siggia [8], as well as other recent contributions to the theory of a passive scalar [2]-[4] and [7], also based on the eddy diffusivity, showed that anomalous scaling can be derived in the eddy diffusivity approximation, provided the zero modes are treated with due respect. (In the problem of a passive scalar advected by a white-in-time random velocity the eddy diffusivity is an exact consequence of the equations of motion).

The scaling exponents, calculated by the present anlysis, are compared with the results of physical experiments (Refs. [14]-[16]) in Table I. The measurement of the two-point correlation functions is very difficult. That is why usually the single-point, two-time correlations functions $F(\tau) =< u(x,t)u(x,t+\tau) >$ are measured and a simple space-time tranformation $r = U\tau$ is used for the interpretation of the data in terms of spatial correlation functions. This transformation is called Taylor frozen turbulence hypothesis, which is accurate when the ratio of the fluctuating $u_{rms}$ and mean $U$ velocities is very large $a = U/u_{rms} >> 1$. This criterion is well satisfied in the wall-bounded flows (boundary layers, pipes, ducts etc ), while in the jets, mixing laers and other open flows $a = O(1)$. The experimental data used in the Table were measured in the atmospheric boundary layer [14] with $a \approx 30$, in the flow between counterrotating disks [15] with $U = 0$ and in the turbulent jet [16]. To account for the space-time relation in an accurate way, the authors of Ref. [15] introduced a Lagrangian-like transformation, based on the idea of the "local Taylor hypothesis", which can be accurate even when $a = 0$. The data of Ref. [16] were collected in the jet flow using a novel nonintrusive optical technique which enables one to measure directly multi-point spatial correlation functions. Due to the experimental uncertainty, we have avoided comparison with the data on the two-time correlation functions, based on the traditional Taylor hypothesis, obtained in the open flows.

The fact that the scaling exponents evaluated in this work agree so well with experimental data is additional evidence that the eddy viscosity approximation is much more powerfull than may have expected for a simple one-loop theory. The eddy viscosity approximation, applied to the inertial range dynamics, accounts for pressure fluctuations only in the numerical value of the factor $\Gamma$ in the eddy viscosity definition [10], [12]. This is not good enough for the correct representation of the effects ifluenced by large-scale velocity fluctuations, which dominate very high-order moments of velocity differences (vortex filaments etc). Moreover, the equation (13) takes into account only the first dissipative anomaly given by (1). It is clear the anomalies, describing constant or close-to-constant fluxes of other flow characteristics, like $K = u_i u_i$, can influence the properties of high- order moments. Since the theory does not account for these effects, it cannot be used for evaluation of the exponents when

$n$ is large enough. To conclude, we would like to mention recent multifractal model by She and Leveque [17], leading to a different expression for the scaling exponents $\xi_n$ which are in a good agreement with experimental data for $n > 0$. The relation between the equation (13) and formula (15), derived here, and the She-Leveque theory is not understood.

I would like to thank K.R.Sreenivasan for communicating his exprerimental findings on the scaling exponents of low-order structure functions prior to publication. Helpful discussions with R.H. Kraichnan, M. Nelkin and S. Orszag are aknowledged. This work was supported in parts by ONR, AFOSR and ARPA.

| n | $\xi_n$,calc | $\xi_n$,[14] | $\xi_n$,[15] | $\xi_n$,[16] |
|---|---|---|---|---|
| 0.1 | 0.0466 | $0.043 \pm 0.006$ | | |
| 0.2 | 0.0913 | $0.083 \pm 0.010$ | | |
| 0.3 | 0.1346 | $0.123 \pm 0.110$ | | |
| 0.5 | 0.2171 | $0.200 \pm 0.150$ | | |
| 1 | 0.4046 | $0.384 \pm 0.023$ | 0.40 | |
| 1.5 | 0.5727 | $0.555 \pm 0.024$ | | |
| 2 | 0.7257 | $0.714 \pm 0.025$ | 0.71 | $0.70 \pm 0.01$ |
| 4 | 1.2433 | 1.21 | 1.24 | $1.28 \pm 0.03$ |
| 5 | 1.4644 | 1.53 | 1.48 | $1.50 \pm 0.05$ |
| 6 | 1.6684 | 1.66 | 1.69 | $1.75 \pm 0.10$ |
| 8 | 2.0378 | 2.05 | | |
| 10 | 2.3690 | 2.38 | | |

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